FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 35, No 4 (2020), 1107–1125
https://doi.org/10.22190/FUMI2004107B

STATISTICAL INFERENCE FOR GEOMETRIC PROCESS WITH THE GENERALIZED RAYLEIGH DISTRIBUTION

Cenker Biçer, Hayrinisa D. Biçer, Mahmut Kara and Asuman Yılmaz

Abstract. In the present paper, the statistical inference problem is considered for the geometric process (GP) by assuming the distribution of the first arrival time with generalized Rayleigh distribution with the parameters α and λ. We have used the maximum likelihood method for obtaining the ratio parameter of the GP and distributional parameters of the generalized Rayleigh distribution. By a series of Monte-Carlo simulations evaluated through the different samples of sizes - small, moderate and large, we have also compared the estimation performances of the maximum likelihood estimators with the other estimators available in the literature such as modified moment, modified L-moment, and modified least squares. Furthermore, wehave presented two real-life datasets analyses to show the modeling behavior of GP with generalized Rayleigh distribution.

Keywords: Monotone processes; non-parametric estimation; parametric estimation; stochastic process; data with trend.

1. Introduction

In 1988, Lam [18] introduced the geometric process (GP) as a simple monotonic stochastic process. In order to model a successive inter-arrival times dataset with a monotone trend, the GP is a quite important alternative to the alpha series process and the nonhomogeneous Poisson process with a monotone intensity function. Since it has a simple form which is easily applied to the many real-life problems from different areas such as science, health, engineering etc., see [17], its popularity increases day by day according to its alternatives. Some key features of the GP and its advantages, which the GP provides in the modeling of the arrival times data with a trend, studied by Lam [16], Lam [18], Lam et al.[19] and Braun et al. [9], [10]. The GP is given by the following definition, see [17].

Received August 29, 2019; accepted April 12, 2020
2020 *Mathematics Subject Classification.* Primary 60G55; Secondary 60K05, 62F12

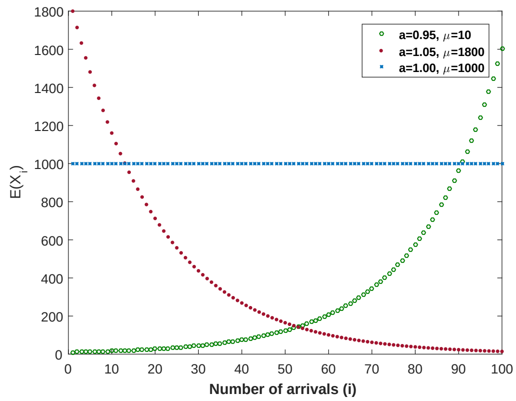

Fig. 1.1: Behavior of the GP

Definition 1.1. Let X_i be the arrival time between the $(i-1)$th and ith events of a counting process $\{N(t), t \geq 0\}$ for $i = 1, 2, ...$ The process $\{X_i, i = 1, ..., n\}$ is said to be a GP with parameter a if there exists a real number $a > 0$ such that $Y_i = a^{i-1}X_i, i = 1, 2, ...,$ are independently and identically distributed (iid) random variables which have any continuous distribution supported on positive real interval. Where a is called the ratio parameter of the GP.

In a general concept, there are three important parameter types in a GP. The first of these parameter types is the ratio parameter a. The second type of them is mean and variance of the first arrival time X_1. In the GP, determining the mean and variance of the first arrival time is quite important because of the fact that the means and variances of the random variables $X_i, i = 1, 2, ...$ are easily represented by the mean and variance of the first arrival time. Assume that $E(X_1) = \mu$ and $Var(X_1) = \sigma^2$ for a GP with the ratio parameter a. By these notations, the mean and variance of the random variable X_i, $(i = 1, 2, \cdots, n)$, are given by following forms:

$$E(X_i) = \frac{\mu}{a^{i-1}} \ , i = 1, 2, ... \tag{1.1}$$

$$Var(X_i) = \frac{\sigma^2}{a^{2(i-1)}}, i = 1, 2, ... \tag{1.2}$$

Hence, by using the relation given by equation 1.1, we can provide Figure 1.1 to illustrate the monotonic behavior of the GP, where the $E(X_i)$ is plotted against the arrival number $i, (i = 1, 2, \cdots,)$ for a fixed μ.

By the Figure 1.1, the process has a monotone increasing behavior when $a < 1$ and has a monotone decreasing behavior when $a > 1$. If $a = 1$ then the process is a Renewal process (RP) [17].

The last type of the important parameters is the distributional parameters of the first occurrence time X_1. In the literature, one can find many published studies related to the parameter estimation problem for both the ratio parameter a and distributional parameters of GP. Lam [16] obtained some non-parametric estimators for parameter a. Several studies that take into account some specific lifetime distributions for first occurrence time X_1 and focus on estimating the distributional parameters of GP are as follows: Gamma [12], Weibull [3], log-normal [18], inverse Gaussian [13], Lindley [7], power Lindley [4], Rayleigh [8], two-parameter Rayleigh [5] and two-parameter Lindley [6] distribution for the GP.

The main motivation of this study is to estimate the parameters of GP when the distribution of first occurrence time is Generalized Rayleigh (GR) also known as two-parameter Burr Type X distribution. We are motivated to the GR distribution for the distribution of the first occurrence time because it is an important alternative to the other famous distributions used in reliability analysis such as the Gamma, Weibull, exponential. In accordance with the purpose of this study, we employ the maximum likelihood (ML), modified moments (MM), modified L-moments (MLM) and modified least-squares (MLS) methods to obtain estimators of the unknown parameters of GP.

The rest of the paper is organized as follows: In section 2, we shall overview the GR distribution. In section 3, we shall obtain the ML estimators of the unknown parameters of GP with the GR distribution. Furthermore, we will investigate some modified estimators for distributional parameters of GP considering the non-parametric estimate of the ratio parameter a. In section 4, some Monte-Carlo simulation studies which compare the efficiencies of the ML estimators obtained in section 3 with the MM, the MLM, and the MLS estimators are performed. Section 5 covers two real-life examples which illustrate the modeling capability of a GP with GR distribution. Section 6 concludes the study.

2. An overview to GR distribution

The GR distribution, also known as two-parameter Burr Type X distribution, was originally studied by Surles and Padgett [22]. Later on, the distribution was renamed as the GR by Raqab and Kundu [21]. The GR is a commonly used probability model in the modeling of positive and non-symmetric data observed from various areas such as communication, health, engineering, reliability etc. Since the distribution is applicable to the modeling of data measured from a wide variety of areas, the interest in the theory and methods related to GR distribution is progressive.

The probability density function (pdf) of the GR distribution with the parameters α and λ is

$$f(x;\alpha,\lambda) = 2\alpha\lambda^2 x e^{-(\lambda x)^2}\left(1-e^{-(\lambda x)^2}\right)^{\alpha-1}, \; x>0, \tag{2.1}$$

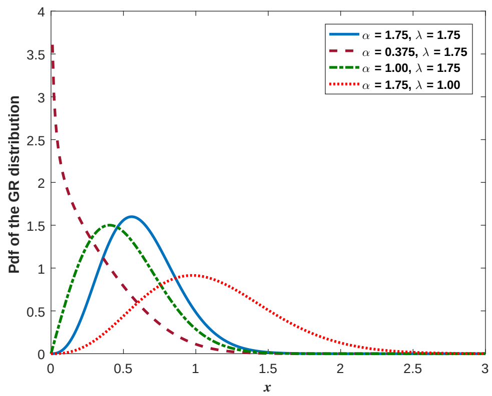

FIG. 2.1: Pdf of the GR distribution for the different values of the parameters

and the corresponding cumulative distribution (cdf) is

$$F(x, \alpha, \lambda) = \left(1 - e^{-(\lambda x)^2}\right)^{\alpha}, \quad x > 0, \tag{2.2}$$

where α and λ are the positive and real valued scale and shape parameters of the distribution, respectively [14]. When $\alpha = 1$, the GR distribution is a Rayleigh with parameter λ. If $\lambda = 1$, then the distribution is reduce to the one-parameter Burr Type X distribution with parameter α. The GR distribution is a unimodal and its pdf is skew to the right when $\alpha > \frac{1}{2}$ and is a decreasing function otherwise [21]. Figure 2.1 below lucidly show the behaviors of the pdf of the GR distribution discussed in here.

The expectation and variance of the GR distribution are not available in the explicit forms, however, they can be easily obtained for selected values of the parameters by using a numeric method.

3. Inference for GP

In this section, in addition to obtaining the ML estimators of the GP with GR distribution, we will also investigate some modified estimators when the ratio parameter of the process is estimated by using a non-parametric estimator.

3.1. ML Estimates

Let us $X_1, X_2, ..., X_n$ be a random sample taken from a GP with ratio a and $X_1 \sim GR(\alpha, \lambda)$ with the pdf (2.1). By considering the equation (2.1) and Definition 1.1, the log-likelihood function for the random variables X_i, $(i = 1, 2, ..., n)$ can be written as

$$\begin{aligned}\ln L(a,\alpha,\lambda) &= n(n-1)\ln a + n\ln 2 + 2n\ln\lambda + n\ln\alpha - \lambda^2 \textstyle\sum_{i=1}^{n}\left(a^{i-1}x_i\right)^2 \\ &\quad + \textstyle\sum_{i=1}^{n}\ln x_i + (\alpha-1)\sum_{i=1}^{n}\ln\left(1-e^{-\left(\lambda a^{i-1}x_i\right)^2}\right).\end{aligned} \tag{3.1}$$

If the first derivatives of Equation (3.1) according to a, α and λ are taken, we have

$$\frac{\partial \ln L(a,\alpha,\lambda)}{\partial a} = \frac{(n-1)n}{a} + 2(\alpha-1)\sum_{i=1}^{n}\frac{(i-1)\lambda^2 a^{2i-3}x_i^2 e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}}{1-e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}} = 0 \tag{3.2}$$

$$\frac{\partial \ln L(a,\alpha,\lambda)}{\partial \lambda} = \frac{2}{\lambda} + (\alpha-1)\sum_{i=1}^{n}\frac{2\lambda a^{2i-2}x_i^2 e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}}{1-e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}} = 0 \tag{3.3}$$

$$\frac{\partial \ln L(a,\alpha,\lambda)}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log\left(1-e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}\right) \tag{3.4}$$

analytical expressions for the ML estimators of the parameters a, λ and α can not be obtained from equations (3.2)-(3.4). However, equations (3.2)-(3.4) can be simultaneously solved using a numerical method such as well-known Newton's method.

Let $\theta = \begin{bmatrix} a \\ \lambda \\ \alpha \end{bmatrix}$ be the parameter vector and likelihood equations given by (3.2)-(3.3) and (3.4) are represented by a gradient vector $\nabla(\theta)$ as

$$\nabla(\theta) = \begin{bmatrix} \frac{\partial \ln L(a,\alpha,\lambda)}{\partial a} \\ \frac{\partial \ln L(a,\alpha,\lambda)}{\partial \lambda} \\ \frac{\partial \ln L(a,\alpha,\lambda)}{\partial \alpha} \end{bmatrix}. \tag{3.5}$$

Thus, in order to estimate of the parameter vector θ, the iterative method given by 3.6 can be used by starting from an initial estimation such as $\hat{\theta}_0$.

$$\theta_{m+1} = \theta_m - H^{-1}(\theta_m)\nabla(\theta_m) \tag{3.6}$$

where $H^{-1}(\theta)$ is the inverse of the Hessian matrix $H(\theta)$.The elements of the matrix $H(\theta)$ are the second derivatives of the log-likelihood function (3.1) with respect to a, λ and α. Let h_{ij} be the $(i,j)\,th$ $(i,j=1,2,3)$ element of the matrix $H(\theta)$. The h_{ij}'s are obtained as below

$$\begin{aligned}h_{11} = &-\frac{(n-1)n}{a^2} + (\alpha-1)\sum_{i=1}^{n}\left(\frac{(2i-3)(2i-2)\lambda^2 a^{2i-4}x_i^2 e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}}{1-e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}}\right. \\ &\left.-\frac{(2i-2)^2\lambda^4 a^{4i-6}x_i^4 e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}}{1-e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}} - \frac{(2i-2)^2\lambda^4 a^{4i-6}x_i^4 e^{-2\lambda^2 a^{2i-2}x_i^2}}{\left(1-e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}\right)^2}\right)\end{aligned} \tag{3.7}$$

$$
\begin{aligned}
h_{12} = \quad & (\alpha-1)\sum_{i=1}^{n}\left(\frac{2(2i-2)\lambda a^{2i-3}x_i^2 e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}}{1-e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}}\right.\\
& \left. -\frac{2(2i-2)\lambda^3 a^{4j-5}x_i^4 e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}}{1-e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}} - \frac{2(2i-2)\lambda^3 a^{4j-5}x_i^4 e^{-2\lambda^2 a^{2i-2}x_i^2}}{\left(1-e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}\right)^2}\right) \qquad (3.8)
\end{aligned}
$$

$$
h_{13} = \sum_{i=1}^{n}\frac{(2i-2)\lambda^2 a^{2i-3}x_i^2 e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}}{1-e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}} \qquad (3.9)
$$

$$
\begin{aligned}
h_{22} = \quad & -\frac{2}{\lambda^2} + (\alpha-1)\sum_{i=1}^{n}\left(\frac{2a^{2i-2}x_i^2 e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}}{1-e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}} - \right.\\
& \left. \frac{4\lambda^2 a^{4i-4}x_i^4 e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}}{1-e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}} - \frac{4\lambda^2 a^{4i-4}x_i^4 e^{-2\lambda^2 a^{2i-2}x_i^2}}{\left(1-e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}\right)^2}\right) \qquad (3.10)
\end{aligned}
$$

$$
h_{23} = \sum_{i=1}^{n}\frac{2\lambda a^{2i-2}x_i^2 e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}}{1-e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}} \qquad (3.11)
$$

$$
h_{33} = -\frac{n}{\alpha^2}. \qquad (3.12)
$$

Note that inverse of the matrix H is calculated as

$$
H^{-1} = \frac{1}{Det\,(H)}\left[\begin{array}{ccc} h_{22}h_{33}-h_{23}h_{32} & -h_{12}h_{33}-h_{13}h_{32} & h_{12}h_{23}-h_{13}h_{22} \\ -h_{21}h_{33}-h_{31}h_{23} & h_{11}h_{33}-h_{13}h_{31} & -h_{11}h_{23}-h_{21}h_{13} \\ h_{21}h_{32}-h_{22}h_{31} & -h_{11}h_{32}-h_{12}h_{31} & h_{11}h_{22}-h_{12}h_{21} \end{array}\right],
$$

where $Det\,(H) = h_{11}h_{22}h_{33} - h_{11}h_{23}h_{32} - h_{12}h_{21}h_{33} + h_{12}h_{31}h_{23} + h_{21}h_{13}h_{32} - h_{13}h_{22}h_{31}$ is determinant of the matrix H. In the Newton method, iterations continue until $\|\theta_{m+1}-\theta_m\| < \varepsilon$ where ε is a predetermined small constant and $\|.\|$ is the Euclidean norm of a vector. Thus, ML estimators of the parameters of GP with GR distribution, say $\hat{a}_{ML}, \hat{\alpha}_{ML}$ and $\hat{\lambda}_{ML}$, are obtained from respective elements of θ_{m+1}.

Now we investigate the asymptotic features of the estimators $\hat{a}_{ML}, \hat{\alpha}_{ML}$ and $\hat{\lambda}_{ML}$. The joint distribution of $\hat{a}_{ML}, \hat{\alpha}_{ML}$ and $\hat{\lambda}_{ML}$ is asymptotic-Normal (AN) with mean vector (a,λ,α) and covariance I^{-1}, where matrix I refers to Fisher information defined as

$$
I = -\frac{1}{n}\left[\begin{array}{ccc} E\left(\frac{\partial \ln L(a,\lambda,\alpha)}{\partial a^2}\right) & E\left(\frac{\partial \ln L(a,\lambda,\alpha)}{\partial a\partial\lambda}\right) & E\left(\frac{\partial \ln L(a,\lambda,\alpha)}{\partial a\partial\alpha}\right) \\ E\left(\frac{\partial \ln L(a,\lambda,\alpha)}{\partial a\partial\lambda}\right) & E\left(\frac{\partial \ln L(a,\lambda,\alpha)}{\partial \lambda^2}\right) & E\left(\frac{\partial \ln L(a,\lambda,\alpha)}{\partial \lambda\partial\alpha}\right) \\ E\left(\frac{\partial \ln L(a,\lambda,\alpha)}{\partial a\partial\alpha}\right) & E\left(\frac{\partial \ln L(a,\lambda,\alpha)}{\partial \lambda\partial\alpha}\right) & E\left(\frac{\partial \ln L(a,\lambda,\alpha)}{\partial \alpha^2}\right) \end{array}\right]. \qquad (3.13)
$$

The elements of the matrix I are written from elements of the Hessian matrix.

3.2. Modified Methods

Lam [16] introduced a non-parametric estimator to estimate only the ratio parameter of the process without making a specific distribution assumption for the GP. The non-parametric estimator of the ratio parameter a is given by, see [16],

$$\hat{a}_{NP} = \exp\left(\frac{6}{(n-1)\,n\,(n+1)}\sum_{i=1}^{n}(n-2i+1)\ln X_i\right). \tag{3.14}$$

The distributional parameters of the GP are easily estimated using the available estimators in the literature when the ratio parameter a is estimated as $\hat{a}_{NP}$. This approximation is known as modified estimation technique in the literature. Now we examine the estimates of the distributional parameters of GP with the GR distribution by assuming that the parameter a is estimated as $\hat{a}_{NP}$. Let $X_1, X_2, ..., X_n$ be a random sample from a GP with ratio a and $X_1 \sim GR(\alpha, \lambda)$, and the parameter a is known as $\hat{a}_{NP}$, from Definition 1.1, we have

$$\hat{Y}_i = \hat{a}_{NP}^{i-1} X_i \tag{3.15}$$

and $\hat{Y}_i \sim GR(\alpha, \lambda)$. Thus, the MM, MLM, and MLS estimators of the α and λ parameters can be obtained as follows by taking into account the moments, L-moments, and least-squares estimators given in [14] and along with the predicted $\hat{Y}_i$.

MM Estimators: The MM estimate of the parameters α, say $\hat{\alpha}_{MM}$ can be obtained from numerical solution of the equation

$$\frac{\psi'(1) - \psi'(\alpha+1)}{(\psi(\alpha+1) - \psi(1))^2} - \frac{V}{U^2} = 0 \tag{3.16}$$

where $U = \frac{1}{n}\sum_{i=1}^{n}\hat{Y}_i^2$, $V = \frac{1}{n}\sum_{i=1}^{n}\hat{Y}_i^4 - U^2$ and $\psi(.)$ is the digamma function, (cf. [1]). Also, by considering $\hat{\alpha}_{MM}$, MM estimates of the parameter λ, say $\hat{\lambda}_{MM}$ is obtained as follows

$$\hat{\lambda}_{MM} = \sqrt{\frac{\psi(\hat{\alpha}_{MM}+1) - \psi(1)}{U}} \tag{3.17}$$

MLM Estimators: The MLM estimates of the parameters α and λ, say $\hat{\alpha}_{MLM}$ and $\hat{\lambda}_{MLM}$, respectively, are obtained by numerical solution of non-linear equation

$$\frac{\psi(2\alpha+1) - \psi(\alpha 1)}{\psi(\alpha+1) - \psi(1)} - \frac{l_2}{l_1} = 0,$$

where $l_1 = \frac{1}{n}\sum_{i=1}^{n}\hat{Y}_{(i)}^2$ and $l_2 = \frac{2}{n(n-1)}\sum_{i=1}^{n}(i-1)\hat{Y}_{(i)}^4 - l_1$ and notation $\hat{Y}_{(i)}$ indicates the ith observation of ordered sample, where $i = 1, 2, ...n$.

MLS Estimators: The MLS estimates of the parameters α and λ, $\hat{\alpha}_{MLS}$ and $\hat{\lambda}_{MLs}$, respectively, are obtained by minimizing the quadratic function $Q(\alpha, \lambda)$

$$Q(\alpha,\lambda)=\sum_{i=1}^{n}\left(\left(1-e^{-\left(\lambda\hat{Y}_{(i)}\right)^2}\right)^{\alpha}-\frac{i}{n+1}\right)^2 \tag{3.18}$$

with respect to α and λ.

For details on deriving these estimators, we refer to [14].

4. Monte-Carlo Simulation Study

In this section, we run some Monte-Carlo simulations to show the estimation performance of ML and modified estimators obtained in the previous section. The main goal of these Monte-Carlo studies, besides displaying the estimation performance of the ML estimators, compare its efficiency with the other estimators. Throughout the Monte-Carlo studies, we set the parameter values as $\lambda = 1$, $\alpha = 0.5, 1$ and 2, and $a = 0.90, 0.95, 1.05, 1.10$. By the 1000 times replicated simulations conducted on the different samples of sizes $n = 30, 50, 100$, we compute the means, biases and $n\times$ mean squared errors ($n\times$MSE) for the ML, MM, MLM and MLS estimates for each collection of parameters. The simulated results are presented in Tables 1-3.

According to the simulation results in Tables 4.1-4.3, we can clearly say that the performances of all estimators are quite satisfactory in all cases. Besides, as the sample size n increases, bias and $n\times$MSE values of all estimators decrease. Thus, we can say that all estimators are asymptotically unbiassed and consistent. In addition, ML estimators outperform the other estimators in small, moderate and large sample sizes.

Table 4.1: The simulated Means, Biases and nxMSEs for the ML, MLS, MM and MLM estimators of the parameters a, α and λ, when $\alpha = 0.5$ and $\lambda = 1$.

			$\hat{a}$			$\hat{\alpha}$			$\hat{\lambda}$		
a	n	Method	Mean	Bias	$n\times$MSE	Mean	Bias	$n\times$MSE	Mean	Bias	$n\times$MSE
0.90	30	ML	0.9021	0.0021	0.0419	0.5526	0.0526	5.5639	1.0555	0.0555	20.2610
		MLS	0.9014	0.0014	0.1107	0.5099	0.0099	4.8498	1.0573	0.0573	52.0585
		MM	0.9014	0.0014	0.1107	0.5675	0.0675	16.1491	1.0724	0.0724	42.9262
		MLM	0.9014	0.0014	0.1107	0.4845	0.0155	6.6621	1.0213	0.0213	36.3018
	50	ML	0.8999	0.0001	0.0123	0.5280	0.0280	2.4407	1.0667	0.0667	16.1984
		MLS	0.9000	0.0000	0.0255	0.5073	0.0073	2.9507	1.0523	0.0523	26.8384
		MM	0.9000	0.0000	0.0255	0.5301	0.0301	8.9010	1.0528	0.0528	27.8249
		MLM	0.9000	0.0000	0.0255	0.4878	0.0122	3.8533	1.0313	0.0313	24.6382
	100	ML	0.8998	0.0002	0.0012	0.5172	0.0172	1.1802	1.0423	0.0423	6.5389
		MLS	0.8996	0.0004	0.0034	0.5057	0.0057	1.4203	1.0512	0.0512	15.3660
		MM	0.8996	0.0004	0.0034	0.5238	0.0238	3.9173	1.0601	0.0601	15.8429
		MLM	0.8996	0.0004	0.0034	0.4981	0.0019	1.5968	1.0451	0.0451	13.9685
0.95	30	ML	0.9507	0.0007	0.0571	0.5590	0.0590	5.5451	1.1014	0.1014	29.9822
		MLS	0.9519	0.0019	0.1450	0.5160	0.0160	6.0655	1.0577	0.0577	52.0999
		MM	0.9519	0.0019	0.1450	0.5943	0.0943	16.8724	1.1040	0.1040	51.0806
		MLM	0.9519	0.0019	0.1450	0.4981	0.0019	6.6786	1.0463	0.0463	44.1838
	50	ML	0.9504	0.0004	0.0107	0.5335	0.0335	3.1944	1.0481	0.0481	14.6898
		MLS	0.9501	0.0001	0.0270	0.5089	0.0089	3.3424	1.0425	0.0425	27.6539
		MM	0.9501	0.0001	0.0270	0.5536	0.0536	9.9358	1.0675	0.0675	30.7541
		MLM	0.9501	0.0001	0.0270	0.4943	0.0057	4.3293	1.0319	0.0319	26.2168
	100	ML	0.9500	0.0000	0.0014	0.5207	0.0207	1.1473	1.0396	0.0396	6.5734
		MLS	0.9501	0.0001	0.0037	0.5094	0.0094	1.3092	1.0314	0.0314	13.4648
		MM	0.9501	0.0001	0.0037	0.5307	0.0307	3.9233	1.0442	0.0442	13.9762
		MLM	0.9501	0.0001	0.0037	0.5037	0.0037	1.4159	1.0291	0.0291	12.2366
1.05	30	ML	1.0508	0.0008	0.0596	0.5529	0.0529	5.1782	1.0974	0.0974	24.8192
		MLS	1.0494	0.0006	0.1488	0.5149	0.0149	6.0798	1.0998	0.0998	57.9221
		MM	1.0494	0.0006	0.1488	0.5785	0.0785	16.3929	1.1198	0.1198	49.7872
		MLM	1.0494	0.0006	0.1488	0.4939	0.0061	6.8040	1.0689	0.0689	40.4968
	50	ML	1.0500	0.0000	0.0140	0.5296	0.0296	2.6421	1.0665	0.0665	13.1142
		MLS	1.0509	0.0009	0.0360	0.5079	0.0079	2.8651	1.0349	0.0349	28.8999
		MM	1.0509	0.0009	0.0360	0.5451	0.0451	8.0653	1.0553	0.0553	28.4833
		MLM	1.0509	0.0009	0.0360	0.4952	0.0048	3.2639	1.0263	0.0263	24.7169
	100	ML	1.0501	0.0001	0.0015	0.5150	0.0150	1.1322	1.0254	0.0254	6.3604
		MLS	1.0502	0.0002	0.0041	0.5037	0.0037	1.3424	1.0147	0.0147	11.7915
		MM	1.0502	0.0002	0.0041	0.5226	0.0226	3.5821	1.0238	0.0238	12.4582
		MLM	1.0502	0.0002	0.0041	0.4954	0.0046	1.4584	1.0079	0.0079	11.0757
1.10	30	ML	1.1017	0.0017	0.0721	0.5657	0.0657	5.3227	1.0710	0.0710	21.8885
		MLS	1.1013	0.0013	0.1583	0.5205	0.0205	4.2594	1.0488	0.0488	41.9511
		MM	1.1013	0.0013	0.1583	0.5946	0.0946	16.2766	1.0973	0.0973	51.3694
		MLM	1.1013	0.0013	0.1583	0.5024	0.0024	5.8000	1.0408	0.0408	40.9797
	50	ML	1.1007	0.0007	0.0155	0.5321	0.0321	2.7774	1.0389	0.0389	12.7543
		MLS	1.1005	0.0005	0.0321	0.5114	0.0114	3.4238	1.0277	0.0277	23.1304
		MM	1.1005	0.0005	0.0321	0.5653	0.0653	10.4034	1.0609	0.0609	24.3047
		MLM	1.1005	0.0005	0.0321	0.5028	0.0028	4.2720	1.0227	0.0227	20.2996
	100	ML	1.1000	0.0000	0.0018	0.5140	0.0140	1.0566	1.0329	0.0329	5.8668
		MLS	1.1003	0.0003	0.0049	0.4964	0.0036	1.2135	1.0060	0.0060	12.3035
		MM	1.1003	0.0003	0.0049	0.5206	0.0206	3.7712	1.0252	0.0252	14.6663
		MLM	1.1003	0.0003	0.0049	0.4928	0.0072	1.4634	1.0076	0.0076	12.4369

Table 4.2: The simulated Means, Biases and nxMSEs for the ML, MLS, MM and MLM estimators of the parameters a, α and λ, when $\alpha = 1$ and $\lambda = 1$.

			$\hat{a}$			$\hat{\alpha}$			$\hat{\lambda}$		
a	n	Method	Mean	Bias	$n\times$MSE	Mean	Bias	$n\times$MSE	Mean	Bias	$n\times$MSE
0.90	30	ML	0.8995	0.0005	0.0226	1.1673	0.1673	34.3936	1.0710	0.0710	12.8072
		MLS	0.9002	0.0002	0.0364	1.0807	0.0807	37.1783	1.0255	0.0255	18.3058
		MM	0.9002	0.0002	0.0364	1.2559	0.2559	84.4448	1.0717	0.0717	18.4970
		MLM	0.9002	0.0002	0.0364	1.0727	0.0727	32.7471	1.0313	0.0313	15.3820
	50	ML	0.9002	0.0002	0.0050	1.0705	0.0705	12.2203	1.0226	0.0226	5.9189
		MLS	0.9006	0.0006	0.0081	1.0206	0.0206	15.2559	0.9911	0.0089	8.0817
		MM	0.9006	0.0006	0.0081	1.1228	0.1228	30.5716	1.0241	0.0241	9.9016
		MLM	0.9006	0.0006	0.0081	1.0159	0.0159	13.6941	0.9961	0.0039	7.9823
	100	ML	0.8999	0.0001	0.0006	1.0335	0.0335	5.5753	1.0233	0.0233	2.8179
		MLS	0.9000	0.0000	0.0009	1.0135	0.0135	7.6169	1.0095	0.0095	4.2659
		MM	0.9000	0.0000	0.0009	1.0668	0.0668	15.4622	1.0268	0.0268	4.8785
		MLM	0.9000	0.0000	0.0009	1.0127	0.0127	7.2168	1.0134	0.0134	3.9930
0.95	30	ML	0.9486	0.0014	0.0302	1.1563	0.1563	32.0993	1.0953	0.0953	14.9887
		MLS	0.9482	0.0018	0.0468	1.0824	0.0824	46.1606	1.0610	0.0610	21.3359
		MM	0.9482	0.0018	0.0468	1.2546	0.2546	75.3895	1.1168	0.1168	21.4723
		MLM	0.9482	0.0018	0.0468	1.0828	0.0828	40.3227	1.0746	0.0746	18.0953
	50	ML	0.9500	0.0000	0.0061	1.0781	0.0781	16.7709	1.0411	0.0411	7.9158
		MLS	0.9499	0.0001	0.0101	1.0414	0.0414	25.0638	1.0266	0.0266	12.0983
		MM	0.9499	0.0001	0.0101	1.1250	0.1250	41.2553	1.0479	0.0479	11.8192
		MLM	0.9499	0.0001	0.0101	1.0263	0.0263	19.4910	1.0258	0.0258	10.3355
	100	ML	0.9500	0.0000	0.0007	1.0217	0.0217	4.4096	1.0115	0.0115	3.0601
		MLS	0.9501	0.0001	0.0012	1.0065	0.0065	6.4943	1.0016	0.0016	5.1477
		MM	0.9501	0.0001	0.0012	1.0593	0.0593	12.7412	1.0177	0.0177	4.8222
		MLM	0.9501	0.0001	0.0012	1.0033	0.0033	6.0221	1.0035	0.0035	4.3940
1.05	30	ML	1.0494	0.0006	0.0265	1.1192	0.1192	21.9043	1.0618	0.0618	9.0571
		MLS	1.0495	0.0005	0.0429	1.0300	0.0300	19.9858	1.0261	0.0261	12.3809
		MM	1.0495	0.0005	0.0429	1.1917	0.1917	60.6033	1.0675	0.0675	14.1094
		MLM	1.0495	0.0005	0.0429	1.0257	0.0257	23.3553	1.0277	0.0277	11.3251
	50	ML	1.0500	0.0000	0.0075	1.0763	0.0763	10.4700	1.0345	0.0345	6.8339
		MLS	1.0497	0.0003	0.0105	1.0247	0.0247	13.3111	1.0186	0.0186	9.5556
		MM	1.0497	0.0003	0.0105	1.1251	0.1251	29.2849	1.0503	0.0503	10.9266
		MLM	1.0497	0.0003	0.0105	1.0229	0.0229	12.6927	1.0246	0.0246	9.1466
	100	ML	1.0502	0.0002	0.0007	1.0364	0.0364	4.7974	1.0122	0.0122	3.0109
		MLS	1.0502	0.0002	0.0013	1.0038	0.0038	5.7562	0.9983	0.0017	4.4178
		MM	1.0502	0.0002	0.0013	1.0683	0.0683	14.2285	1.0192	0.0192	5.3255
		MLM	1.0502	0.0002	0.0013	1.0061	0.0061	6.1276	1.0032	0.0032	4.4140
1.10	30	ML	1.1003	0.0003	0.0353	1.1368	0.1368	27.3646	1.0665	0.0665	12.3611
		MLS	1.0998	0.0002	0.0460	1.0416	0.0416	30.0485	1.0269	0.0269	15.1012
		MM	1.0998	0.0002	0.0460	1.2349	0.2349	70.7584	1.0873	0.0873	17.8739
		MLM	1.0998	0.0002	0.0460	1.0502	0.0502	29.7390	1.0422	0.0422	14.3413
	50	ML	1.0999	0.0001	0.0075	1.1114	0.1114	15.9703	1.0530	0.0530	6.8821
		MLS	1.1001	0.0001	0.0113	1.0520	0.0520	17.7979	1.0204	0.0204	9.3748
		MM	1.1001	0.0001	0.0113	1.1819	0.1819	39.7175	1.0613	0.0613	10.0945
		MLM	1.1001	0.0001	0.0113	1.0592	0.0592	17.1944	1.0308	0.0308	8.3747
	100	ML	1.1001	0.0001	0.0010	1.0332	0.0332	4.9204	1.0139	0.0139	3.3641
		MLS	1.1000	0.0000	0.0017	1.0073	0.0073	6.3576	1.0052	0.0052	5.1185
		MM	1.1000	0.0000	0.0017	1.0602	0.0602	15.3053	1.0208	0.0208	5.3203
		MLM	1.1000	0.0000	0.0017	1.0062	0.0062	6.4213	1.0080	0.0080	4.5479

Table 4.3: The simulated Means, Biases and nxMSEs for the ML, MLS, MM and MLM estimators of the parameters a, α and λ, when $\alpha = 2$ and $\lambda = 1$.

			$\hat{a}$			$\hat{\alpha}$			$\hat{\lambda}$		
a	n	Method	Mean	Bias	$n\times$MSE	Mean	Bias	$n\times$MSE	Mean	Bias	$n\times$MSE
0.90	30	ML	0.8999	0.0001	0.0134	2.3764	0.3764	185.0170	1.0691	0.0691	8.3586
		MLS	0.9001	0.0001	0.0172	2.2418	0.2418	399.8574	1.0313	0.0313	8.9383
		MM	0.9001	0.0001	0.0172	2.3288	0.3288	116.3478	1.0636	0.0636	9.1816
		MLM	0.9001	0.0001	0.0172	2.2279	0.2279	308.3195	1.0429	0.0429	8.3048
	50	ML	0.8996	0.0004	0.0031	2.1643	0.1643	70.3287	1.0346	0.0346	4.9931
		MLS	0.8997	0.0003	0.0037	2.0955	0.0955	105.0502	1.0178	0.0178	5.8255
		MM	0.8997	0.0003	0.0037	2.1857	0.1857	92.2488	1.0319	0.0319	5.8339
		MLM	0.8997	0.0003	0.0037	2.0743	0.0743	84.8229	1.0191	0.0191	5.3602
	100	ML	0.9000	0.0000	0.0003	2.0767	0.0767	22.3545	1.0111	0.0111	1.6830
		MLS	0.9000	0.0000	0.0004	2.0189	0.0189	29.2812	1.0009	0.0009	2.1476
		MM	0.9000	0.0000	0.0004	2.1333	0.1333	48.6809	1.0162	0.0162	2.3114
		MLM	0.9000	0.0000	0.0004	2.0183	0.0183	26.2329	1.0037	0.0037	1.8638
0.95	30	ML	0.9503	0.0003	0.0124	2.3595	0.3595	168.6435	1.0486	0.0486	7.1311
		MLS	0.9502	0.0002	0.0153	2.1344	0.1344	159.8323	1.0111	0.0111	8.0305
		MM	0.9502	0.0002	0.0153	2.3240	0.3240	120.3577	1.0480	0.0480	7.8498
		MLM	0.9502	0.0002	0.0153	2.1655	0.1655	146.1292	1.0262	0.0262	7.5533
	50	ML	0.9498	0.0002	0.0030	2.1902	0.1902	74.3277	1.0317	0.0317	4.1722
		MLS	0.9498	0.0002	0.0035	2.0753	0.0753	84.4408	1.0100	0.0100	4.7112
		MM	0.9498	0.0002	0.0035	2.2398	0.2398	83.9141	1.0374	0.0374	4.8533
		MLM	0.9498	0.0002	0.0035	2.0776	0.0776	70.6279	1.0171	0.0171	4.3557
	100	ML	0.9501	0.0001	0.0004	2.1205	0.1205	38.9492	1.0084	0.0084	1.9344
		MLS	0.9501	0.0001	0.0005	2.0379	0.0379	40.9310	0.9959	0.0041	2.3678
		MM	0.9501	0.0001	0.0005	2.1333	0.1333	62.1434	1.0075	0.0075	2.6137
		MLM	0.9501	0.0001	0.0005	2.0401	0.0401	38.3668	0.9983	0.0017	2.2154
1.05	30	ML	1.0503	0.0003	0.0185	2.2911	0.2911	164.7333	1.0347	0.0347	7.3818
		MLS	1.0505	0.0005	0.0218	2.0555	0.0555	211.5846	0.9842	0.0158	8.1717
		MM	1.0505	0.0005	0.0218	2.2707	0.2707	129.7693	1.0283	0.0283	8.5876
		MLM	1.0505	0.0005	0.0218	2.0674	0.0674	141.8400	1.0018	0.0018	7.7984
	50	ML	1.0499	0.0001	0.0032	2.1497	0.1497	67.0813	1.0352	0.0352	3.4191
		MLS	1.0500	0.0000	0.0039	2.0691	0.0691	97.2026	1.0112	0.0112	4.3589
		MM	1.0500	0.0000	0.0039	2.2045	0.2045	82.0182	1.0357	0.0357	4.0980
		MLM	1.0500	0.0000	0.0039	2.0698	0.0698	87.2862	1.0183	0.0183	3.7935
	100	ML	1.0500	0.0000	0.0005	2.0791	0.0791	26.5465	1.0111	0.0111	2.1213
		MLS	1.0500	0.0000	0.0007	2.0259	0.0259	39.8430	1.0009	0.0009	2.6829
		MM	1.0500	0.0000	0.0007	2.1326	0.1326	55.8938	1.0161	0.0161	2.8456
		MLM	1.0500	0.0000	0.0007	2.0264	0.0264	33.9811	1.0047	0.0047	2.4850
1.10	30	ML	1.0998	0.0002	0.0174	2.3212	0.3212	194.9628	1.0429	0.0429	6.9826
		MLS	1.0997	0.0003	0.0214	2.1402	0.1402	165.2003	1.0096	0.0096	7.5882
		MM	1.0997	0.0003	0.0214	2.3034	0.3034	121.5048	1.0441	0.0441	8.2323
		MLM	1.0997	0.0003	0.0214	2.1614	0.1614	156.1144	1.0235	0.0235	7.5732
	50	ML	1.1003	0.0003	0.0034	2.1643	0.1643	79.7157	1.0159	0.0159	2.9575
		MLS	1.1001	0.0001	0.0046	2.1007	0.1007	110.5767	1.0066	0.0066	4.3802
		MM	1.1001	0.0001	0.0046	2.1697	0.1697	97.4672	1.0176	0.0176	4.0407
		MLM	1.1001	0.0001	0.0046	2.0696	0.0696	90.7552	1.0067	0.0067	3.7026
	100	ML	1.0997	0.0003	0.0006	2.1008	0.1008	30.4018	1.0329	0.0329	2.0447
		MLS	1.0997	0.0003	0.0007	2.0582	0.0582	40.5492	1.0243	0.0243	2.5109
		MM	1.0997	0.0003	0.0007	2.1426	0.1426	50.4873	1.0366	0.0366	2.8699
		MLM	1.0997	0.0003	0.0007	2.0540	0.0540	36.9373	1.0263	0.0263	2.4028

5. Application

In this section, we analyze two real-life datasets called No.3 data and Software data to illustrate the estimation procedures the ML, the MM, the MLM and the MLS. To compare the RP and GPs with the ML, the MM, the MLM and the MLS estimators, we use the mean-squared error (MSE*) criterion defined as, see [?],

- MSE*$= (1/n) \sum_{k=1}^{n} \left(X_k - \hat{X}_k \right)^2$,

where $\hat{X}_k$ is calculated by

$$\hat{X}_k = \begin{cases} \hat{\mu}_{(ML)} \hat{a}_{ML}^{1-k} & \text{GP with the ML estimators,} \\ \hat{\mu}_{(MLS)} \hat{a}_{NP}^{1-k} & \text{GP with the MLS estimators,} \\ \hat{\mu}_{(MM)} \hat{a}_{NP}^{1-k} & \text{GP with the MM estimators,} \\ \hat{\mu}_{(MLM)} \hat{a}_{NP}^{1-k} & \text{GP with the MLM estimators,} \\ \hat{\mu}_{(ML)} & \text{RP with the ML estimators,} \end{cases} \tag{5.1}$$

and $\hat{\mu}_{(.)}$ is estimate of the expected value of the first occurrence time under the fitted GR distribution with the ML, MM, MLM and MLS estimators and can be numerically calculated from

$$\hat{\mu}_{(.)} = \int_0^\infty x f \left(x, \hat{\alpha}_{(.)}, \hat{\lambda}_{(.)} \right) dx$$

No.3 data:

In the No.3 data set, there are 71 observations, which are regarding the unscheduled maintenance actions for U.S.S. Halfbeak No.3 main propulsion diesel engine [2]. This data set was found to be consistent with a GP in which the ratio parameter is greater than 1, see [16].

In the first stage of data analysis, we investigate whether the data set follows a GR distribution. Linear regression model

$$\ln X_i = \tau - (i-1) \ln a + \varepsilon_i \tag{5.2}$$

can be employed to this aim, see [13] for further information on derivation of this regression model. Where $\tau = E(\ln Y_i)$, $Y_i = a^{i-1} X_i$ and $\exp(\varepsilon_i) \sim GR(\theta, \beta)$. The error term ε_i given in equation (5.2) can be easily estimated by

$$\hat{\varepsilon}_i = \ln X_i - \hat{\tau} - (i-1) \ln \hat{a}_{NP} \tag{5.3}$$

where $\hat{\tau} = \frac{n(n-1)}{2} \ln \hat{a}_{NP} + \sum_{i=1}^{n} \ln X_i$. Thus, we can say that the data set is consistent with a GR distribution if the exponential errors follow a GR distribution. The parameters estimations of the exponential errors are $\hat{\theta}_{ML} = 0.2410$ and

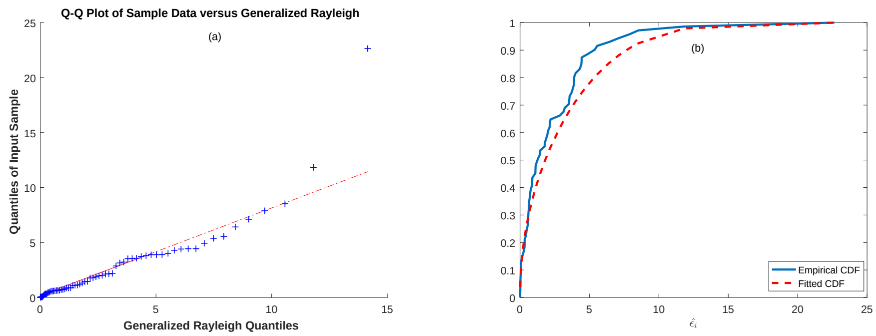

FIG. 5.1: QQ plot for the exponential errors (a), empirical and fitted cdf for the exponential errors (b)

Table 5.1: Estimation of parameters for the No 3 data set

Process	Method	$\hat{a}$	$\hat{\alpha}$	$\hat{\lambda}$	MSE/10^5
GP	ML	1.04272	0.12795	0.0002	1.93257
	MLS		0.2596	0.0004	2.0208
	MM	1.0416	0.0700	0.0002	2.2717
	MLM		0.1277	0.0002	2.0210
RP	ML	1.0000	0.1910	0.0007	3.3945

$\hat{\beta}_{ML} = 0.1330$ and also the value of Kolmogorov-Smirnov (K-S) test is 0.1286 and corresponding p-value is 0.1751. Hence, result of the K-S test, we can say that the No. 3 dataset consistent with a GR distribution. To confirm this result, we present Figure 5.1(a) and Figure 5.1 (b). Figure 5.1(a) displays the Q-Q plot of quantiles of the data versus $GR\,(\theta, \beta)$. Figure 5.1 (b) display both the empirical and fitted cdf. As it can be clearly seen from Figure 5.1 (a), the quantiles of the data fall approximately on the straight line. In Figure 5.1 (b), the fitted cdf closely follows to empirical cdf.

If the GP with the GR is applied to this data, the parameter estimates obtained by using the employed estimators in the paper and the corresponding MSE values are presented in Table 5.1

From Table 5.1, it is seen that the GP outperform the RP for this data set. Besides, the GP with ML estimators have the lowest MSE value relative to other models. We present the Figure 5.2 to show the relative performances of the four GPs with the ML, the MM, the MLM and the MLS estimators and the RP. Figure 5.2 display the plots of S_k, $S_k = X_1 + X_2 + ... + X_k$, $k = 1, 2, ..., n$ and its estimates $\hat{S}_k$, $\hat{S}_k = \sum_{j=1}^{k} \hat{X}_k$, against the k, $k = 1, 2, ..., n$, where $\hat{X}_k$ can obtained by using (5.1).

According to Figure 5.2, it can be concluded that GPs follow true values more accurately than RP.

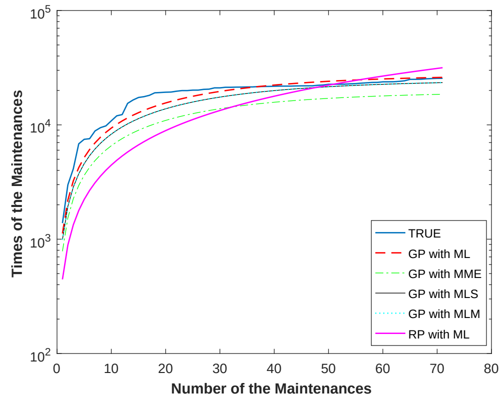

FIG. 5.2: The plots of the observed and estimated maintenance times for the No. 3 data set

Table 5.2: Estimates and evaluated MSE* values of the different GP models for the No. 3 data

	Model									
	G. Rayleigh		Gamma		Log-Normal		Weibull		Inv. Gaussian	
MSE*/10^5	1.93257		2.15623		2.46508		2.11300		1.93442	
Parameter Est.	$\hat{a}$	1.04272	$\hat{a}$	1.03547	$\hat{a}$	1.04165	$\hat{a}$	1.03659	$\hat{a}$	1.04274
	$\hat{\alpha}$	0.12795	$\hat{k}_G$	0.66991	$\hat{\mu}_{LN}$	6.06255	$\hat{\theta}_W$	777.7413	$\hat{\mu}_{IG}$	1118.4
	$\hat{\lambda}$	0.0002	$\hat{\theta}_G$	1290.572	$\hat{\sigma}_{LN}$	1.68506	$\hat{\lambda}_W$	0.7730	$\hat{\sigma}_{IG}$	1781.1

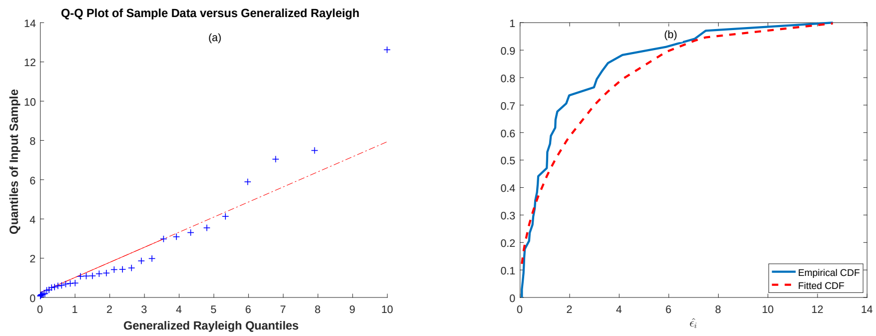

FIG. 5.3: QQ plot for the exponential errors (a), emprical and fitted cdf for the exponential errors (b)

Table 5.3: Estimation of parameters for the software data

Process	Method	$\hat{a}$	$\hat{\alpha}$	$\hat{\lambda}$	MSE/10^3
GP	ML	0.9094	0.3108	0.1319	1.8027
	MLS		0.3032	0.1023	2.1646
	MM	0.9370	0.1352	0.0493	2.0867
	MLM		0.1293	0.0483	2.0965
RP	ML	1.0000	0.1845	0.0087	2.6559

Software data:

Software data set includes 34 observations. These data represent the time between successive failures of a piece of software developed as part of a large data system [11]. Braun et al. [9] showed that this data set consistent by a GP with the ratio parameter $a < 1$. Thus we can apply a GP with the GR distribution to this data. First, we investigate whether the underlying distribution of the data is consistent with a GR distribution, as in the No. 3 data. When the regression given by (5.2) is applied to this data, estimates of the parameters for the exponential errors are $\hat{\theta}_{ML} = 0.2381$ and $\hat{\beta}_{ML} = 0.1677$. For this data, K-S test is 0.1615 and corresponding p-value is 0.3040. Thus, we can say that the software data set consistent with a GR distribution. In addition, we present the Q-Q plot and the fitted and empirical cdf of the exponential errors by the Figure 5.3 to support the result of K-S test.

When a GP with the GR distribution is applied to software data set, estimates of the parameters a, α and λ and the corresponding MSE values are given in Table 5.3

Acording to Table 5.3, GP outperform the RP since it has lower MSE. Furthermore, GP with the ML estimates has the best performance among all GPs. Furthermore, relative performances of the GPs with the all estimators and RP can be seen from Figure 5.4. Figure 5.4 include the plots of the S_k and $\hat{S}_k$'s against the

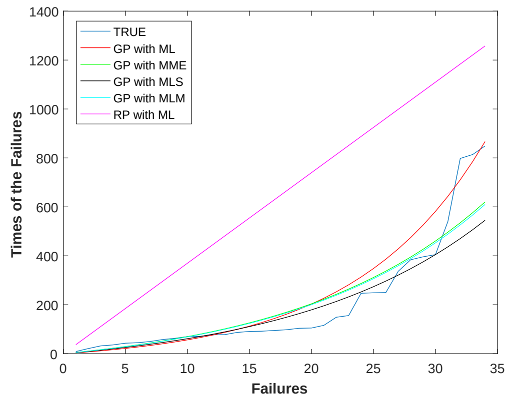

FIG. 5.4: The plots of the observed and estimated failure times for the Software data

Table 5.4: Estimates and evaluated MSE* values of the different GP models for the Software data

	Model									
	G. Rayleigh		Gamma		Log-Normal		Weibull		Inv. Gaussian	
MSE*/10^3	1.8027		1.8763		1.9887		1.8890		2.1314	
Parameter Est.	$\hat{a}$	0.9094	$\hat{a}$	0.9172	$\hat{a}$	0.9370	$\hat{a}$	0.9186	$\hat{a}$	0.9504
	$\hat{\alpha}$	0.3108	$\hat{k}_G$	0.8533	$\hat{\mu}_{LN}$	1.0017	$\hat{\theta}_W$	3.6726	$\hat{\mu}_{IG}$	7.5144
	$\hat{\lambda}$	0.1319	$\hat{\theta}_G$	4.4649	$\hat{\sigma}_{LN}$	1.2742	$\hat{\lambda}_W$	0.8856	$\hat{\sigma}_{IG}$	14.4192

k, $k = 1, 2, ..., n$, where S_k and $\hat{S}_k$ are defined as in the previous example.

As in the previous example, we can easily seen from Figure 5.4 that four GPs follow true values more accurately than RP.

6. Conclusion

The GP with the GR distribution considered by this article has many potential uses for modeling of successive arrival times observed from many fields. The process is very suitable for modeling applications of arrival times with the monotonic ascending or descending behavior as highlighted in the paper. The monotonic behavior of the GP is controlled by a positive-valued ratio parameter a, which is an essential feature of this process. In the paper, for the different values of the parameter a, the behavior of the process has been clearly illustrated in Figure 1.1. In addition to the ratio parameter a, the parameters of the distribution of the first arrival time are other key parameters that regulate the behavior of the process. In order to achieve an optimal modeling performance from the GP, the solution of the estimation problem of these parameters is crucial. The estimation problem for a, α and λ parameters of GP with the GR distribution is solved by employing the

ML methodology in the paper. The results of numerical studies which compare the efficiency of the ML estimators and modified estimators considered in this paper are presented in the tables. Tabulated results display that the ML estimators produce more efficient estimations in all cases with respect to bias and MSE criterion.

In order to demonstrate the phases of data modeling by a GP with the GR distribution and comparing its modeling performance against the RP, in the paper, two examples are carried out on real-world datasets called the No.3 and Software. In both examples, the GP with the GR distribution outperforms the RP with smaller MSE values. Furthermore, by the analysis of the results in the paper, it can be concluded that fitting by a GP with the GR distribution to both data sets is better than fitting by a GP with the possible alternatives of the GR distribution such as Gamma, Log-Normal, inverse Gaussian and Weibull.

7. Acknowledgments

The author thanks the anonymous referees for their comments and suggestions for improving the first version of this paper.

REFERENCES

1. M. Abramowitz and I. A. Irene: *Handbook of mathematical functions: with formulas, graphs, and mathematical tables*. Courier Corporation, 1964
2. H. Ascher and H. Feingold: *Repairable Systems Reliability*. Marcel Dekker, New York, 1984
3. H. Aydoğdu and B. Şenoğlu and M. Kara: *Parameter estimation in geometric process with Weibull distribution*. Applied Mathematics and Computation. **217(6)** (2010), 2657–2665.
4. C. Biçer: *Statistical Inference for Geometric Process with the Power Lindley Distribution*. Entropy, **20(10)**, 2018, 723.
5. C. Biçer: *Statistical inference for geometric process with the Two-parameter Rayleigh Distribution*. The Most Recent Studies in Science and Art, **1**, (2018), 576–583.
6. H. D. Biçer: *Statistical inference for geometric process with the Two-Parameter Lindley Distribution*. Communications in Statistics-Simulation and Computation, (2019), 1–22.
7. C. Biçer and H. D. Biçer: *Statistical Inference for Geometric Process with the Lindley Distribution*. Researches on Science and Art in 21st Century Turkey, **2**, (2017), 2821–2829.
8. C. Biçer, and H. D. Biçer and M. Kara and H. Aydoğdu, Halil: *Statistical inference for geometric process with the Rayleigh distribution*. Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, **68(1)**, 2019, 149–160.
9. W. J. Braun, and W. Li and Y. Q. Zhao: *Properties of the geometric and related processes*. Naval Research Logistics. **52(7)**, (2005), 607–616.

10. W. J. Braun, and W. Li and Y. Q. Zhao: *Some theoretical properties of the geometric and α-series processes.*Communications in Statistics Theory and Methods. **37(9)**, (2008), 1483-1496.

11. M. J. Crowder and A. C. Kimber and R. L. Smith and T. J. Sweeting: *Statistical concepts in reliability*. Chapman and Hall, London, 1991.

12. J. S. K. Chan and, Y. Lam and D. Y. P. Leung: *Statistical inference for geometric processes with gamma distributions*. Computational statistics & data analysis. **47(3)**, (2004), 565–581.

13. M. Kara and H. Aydoğdu and Ö. Türkşen: *Statistical inference for geometric process with the inverse Gaussian distribution*. Journal of Statistical Computation and Simulation. **85(16)**, (2015), 3206–3215.

14. D. Kundu and M. Z. Raqab: *Generalized Rayleigh distribution: different methods of estimations*. Computational statistics & data analysis. **49(1)**, (2005), 187–200.

15. Y. Lam: *A note on the optimal replacement problem*. Advances in Applied Probability. **20(2)**, (1988), 479–482.

16. Y. Lam: *Nonparametric inference for geometric processes*. Communications in statistics-theory and methods. **21(7)**, 1992, 2083–2105.

17. Y. Lam: *The geometric process and its applications*. World Scientific, 2007.

18. Y. Lam and S. K. Chan: *Statistical inference for geometric processes with lognormal distribution*. Computational statistics & data analysis. **27(1)**, (1998), 99–112.

19. Y. Lam and Y. Zheng and Y. Zhang: *Some limit theorems in geometric processes*. Acta Mathematicae Applicatae Sinica, English Series. **19(3)**, (2003), 405-416.

20. Y. Lam and L. Zhu and J. S. K. Chan and Q. Liu: *Analysis of data from a series of events by a geometric process model*. Acta Mathematicae Applicatae Sinica, English Series. **20(2)**, (2004), 263–282.

21. M. Z. Raqab and D. Kundu: *Burr type X distribution: revisited*. Journal of probability and statistical sciences. **4(2)**, (2006), 179–193.

22. J.G. Surles and W. J. Padgett: *Inference for reliability and stress-strength for a scaled Burr type X distribution*. Lifetime Data Analysis, **7(2)**, (2001), 187–200.

Cenker Biçer
Faculty of Arts and Sciences
Department of Statistics
The University of Kırıkkale
71450 Kırıkkale, Turkey
cbicer@kku.edu.tr

Hayrinisa Demirci Biçer
Faculty of Arts and Sciences
Department of Statistics
The University of Kırıkkale
71450 Kırıkkale, Turkey
hdbicer@kku.edu.tr

Mahmut Kara
Faculty of Economics and Administrative Sciences
Department of Econometrics
The University of Yüzüncü Yıl
Van, Turkey
mkara2581@gmail.com

Asuman Yılmaz
Faculty of Economics and Administrative Sciences
Department of Econometrics
The University of Yüzüncü Yıl
Van, Turkey
asumanduva@gmail.com